Bifractality of the Devil's staircase appearing in the Burgers equation with Brownian initial velocity

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Abstract

It is shown that the inverse Lagrangian map for the solution of the Burgers equation (in the inviscid limit) with Brownian initial velocity presents a bifractality (phase transition) similar to that of the Devil's staircase for the standard triadic Cantor set. Both heuristic and rigorous derivations are given. It is explained why artifacts can easily mask this phenomenon in numerical simulations.

1 Introduction

There is a renewed interest among physicists and mathematicians in the Burgers equation

$$\partial_t v + v\partial_x v = \nu \partial_x^2 v. \tag{1}$$

In particular it has been discovered that, when the initial velocity $v_0(x)$ is a Brownian (or fractional Brownian) motion function of the space coordinate x and the limit of vanishing viscosity ν is taken, the Lagrangian map

$$a \mapsto x(a, t) \tag{2}$$

from the initial fluid particle position a to its position x at time t is a Devil's staircase [1, 2, 3]; this has consequences for the distributions of masses of large-scale structures in the Universe [3].

Since the Lagrangian map (2) is nondecreasing (a consequence of the fact that fluid particles can merge but not cross), two nonnegative measures may be defined via their increments. The direct Lagrangian measure associates to a Lagrangian interval $[a, b]$ the length $\Delta x = x(b) - x(a)$. The inverse Lagrangian measure associates to an Eulerian interval $[x, y]$ the length $\Delta a = a(y) - a(x)$. When the initial density field is

uniform, this length Δa is proportional to the mass in the interval and will therefore be called the "mass". The multifractal properties of the direct and inverse Lagrangian measures can be analyzed by studying the scaling properties of the moments of their increments. Here, we shall be interested only in the inverse Lagrangian measure, whose moments are defined by

$$M_q(l) \equiv \langle [a(x+l) - a(x)]^q \rangle, \qquad q \geq 0, \tag{3}$$

which does not depend on x because the Lagrangian map has homogeneous (stationary in the space variable) increments.

In Ref. [2] it was conjectured that the inverse Lagrangian measure has a bifractality similar to that known for the standard Devil's staircase associated to the triadic Cantor set (see Fig. 1). According to the conjecture, the scaling exponents τ_q, obtained from the small-l behavior of the moments

$$M_q(l) \propto l^{\tau_q}, \tag{4}$$

should present a *phase transition*: for $q \geq q_\star$, one should have $\tau_q = 1$, while for $0 \leq q \leq q_\star$, one should have $\tau_q = q/q_\star$. Phase transitions of this sort are frequently observed in studying fractal sets of physical interest (see, e.g., Ref.[4]). Preliminary numerical tests, reported in Refs. [2, 3] were rather inconclusive. Further (unpublished) simulations indicated the presence of a small-l scaling régime with $\tau_q = 1$ for any q. While developing the correct theory for the inverse Lagrangian map, we found that this is actually an artifact inherent to discrete numerical simulations which can hide the true scaling.

Here, we demonstrate the conjectured bifractality. Section 2 is devoted to recalling some known results about the solution of the Burgers equation and to formulating the problem with (fractional) Brownian initial conditions. In Section 3 we present a heuristic approach (the physicist's viewpoint); the arguments are not rigorous, but encompass both the Brownian and the fractional Brownian case; for pedagogical reasons, the arguments are presented first for the standard Devil's staircase. In Section 4, we present a rigorous proof for the Brownian case. In Section 5, we present numerical simulations, discuss the aforementioned artifact and show how to overcome it.

2 The Lagrangian map for the Burgers equation

We recall the construction of the solution to (1) (see Refs. [1, 2, 3] for details). The solution at time t, for a continuous initial velocity $v_0(x)$, is given by

$$v(x,t) = v_0(a(x,t)). \tag{5}$$

Here, $x \mapsto a(x,t)$ is the inverse Lagrangian map obtained as follows: we denote by $\psi_0(x)$ the initial velocity potential ($v(x) = -\partial_x \psi(x)$); then the Lagrangian location

$a(x,t)$ is such that $\psi_0(a) - (x-a)^2/(2t)$ achieves its global maximum. This map is nondecreasing and has discontinuities at shock locations. Its inverse $a \mapsto x(a,t)$ is called the Lagrangian map. Those Lagrangian locations which are not mapped into shocks are called regular.

For the case of fractional Brownian initial velocity, $v_0(x)$ is a random function, defined from $-\infty$ to $+\infty$, which is Gaussian and satisfies the following conditions (angular brackets denote averages):

$$\begin{aligned} \langle v_0(x) \rangle &= 0, & (6) \\ \langle [v_0(x') - v_0(x)]^2 \rangle &= C^2 |x'-x|^{2h}, & (7) \end{aligned}$$

with $0 < h < 1$. The case $h = 1/2$ is the standard Brownian motion curve (in the space variable) for which it was shown in Ref. [1] that the regular points form (almost surely) a set of Hausdorff dimension $D = 1/2$. For the case $h \neq 1/2$, very strong numerical evidence was given in Ref. [3] that $D = h$. It follows that the Lagrangian map is a Devil's staircase.

The scaling properties of the fractional Brownian motion $v_0(x)$ and of the Burgers equation imply that [2, 3]

$$v(x,t) \stackrel{\text{law}}{=} t^{\frac{h}{1-h}} v\left(xt^{-\frac{1}{1-h}}, 1\right), \tag{8}$$

where $\stackrel{\text{law}}{=}$ denotes probabilistic equality in law. Hence, the knowledge of the statistical properties at time $t = 1$ gives those at any other $t > 0$ by simple rescaling. Henceforth, in discussing the theory, we shall sometimes set $t = 1$ without loss of generality. In performing numerical simulations with a finite mesh and a maximum box size L, the choice of the time t becomes relevant since the only scales which can be meaningfully related to the untruncated problem are between the mesh-size and L.[1] By changing t we can adjust the coalescence length

$$l_c(t) \equiv (Ct)^{1/(1-h)} \tag{9}$$

so that it does not lie too close to either of these. Over separations $l \gg l_c(t)$ the velocity increments, as well as the Lagrangian measures, remain basically unaffected, because not enough time has elapsed for substantial particle merging. In other words, the Lagrangian map is close to the identity.

3 A heuristic approach

We begin by recalling in some detail how bifractality arises for the standard inverse Devil's staircase (Fig. 1). Let a denote the coordinate on the vertical (Lagrangian)

[1] Or a subset thereof, as we shall see in Section 5.

axis and x the coordinate on the horizontal (Eulerian) axis. At all (dyadic) x's which are integer multiples of 2^{-n} there is a jump (shock) of the inverse Lagrangian map $a(x)$. It is straightforward to show that there are

$$\begin{gathered}2^0 \text{jumps of amplitude } 3^{-1}\\ 2^1 \text{ jumps of amplitude } 3^{-2}\\ \dots\\ 2^n \text{ jumps of amplitude } 3^{-(n+1)}\\ \dots\end{gathered} \tag{10}$$

(This formula, which gives the distribution of the shock amplitudes, will be referred to as the mass distribution.)

What is the equivalent of the quantity $M_q(l)$ defined above? First, we shall restrict ourselves to increments l between two successive dyadic points $x_{n,p} = p\,2^{-n}$ and $x_{n,p+1}$ of the same nth generation, so that $l = 2^{-n}$. The average $\langle Q\rangle$ of a quantity Q will be just a sum over the 2^n intervals of this type, divided by 2^n. For example,

$$M_q(l) = \langle [a(x+l) - a(x)]^q \rangle \tag{11}$$

will be evaluated for $l = 2^{-n}$ as

$$M_q(l) = (1/2^n) \sum_{p=0}^{p=2^n-1} [a(x_{n,p+1}) - a(x_{n,p})]^q. \tag{12}$$

Now, an important *remark*: If we do not restrict the Eulerian positions to dyadic points $x_{n,p}$, or if we work with a randomized version of the Devil's staircase, the Lagrangian increment $a(x') - a(x)$ will be a sum over all the Lagrangian (shock) intervals corresponding to the dyadic Eulerian points between x and x'. This will generally include infinitely many intervals of generations $n' \geq n$. However, since the number of such intervals grows as $2^{n'}$, while their length decreases as $3^{-n'}$, the sum is dominated by just the first few terms for which n' is equal to n, or just a bit larger. Hence, we can replace the Lagrangian increment over an Eulerian distance $l = 2^{-n}$ by just the contribution coming from the nth generation, thereby committing an error which just affects constants but not scaling. We thus have

$$M_q(l) \sim (1/2^n) \sum_{p=0}^{p=2^n-1} [\text{length of } p\text{th Lagrangian interval of } n\text{th generation}]^q. \tag{13}$$

The mass distribution gives us the number of shock intervals of a given length. Summing over all the generations $m \leq n$, we obtain :

$$M_q(l) \sim (1/2^n) \sum_{m=0}^{m=n} 2^m [3^{-(m+1)}]^q. \tag{14}$$

It is clear that we must distinguish two cases :
(i) when $q > D = \ln 2/\ln 3$, the sum is dominated by its first term and we obtain

$$M_q(l) \sim (1/2^n) \sim l^1, \tag{15}$$

so that $\tau_q = 1$;
(ii) when $0 \leq q < D$, the sum is dominated by its last term and we obtain

$$M_q(l) \sim (1/2^n)2^n[3^{-(n+1)}]^q \sim 3^{-nq} \sim l^{q\ln 3/\ln 2}, \tag{16}$$

so that $\tau_q = q\ln 3/\ln 2$, which establishes the bifractality for the standard Devil's staircase. This is seen as a rather elementary result, obtained without recourse to numerical computations, unlike the much more intricate phase transitions encountered in the study of certain circle maps [4].

We now turn to the Burgers problem for arbitrary $0 < h < 1$. The mass function is known [1, 3]: the mean number per unit length of Lagrangian intervals with a length between Δa and $\Delta a/2$ is

$$N(\Delta a) \propto (\Delta a)^{-h}. \tag{17}$$

Sinai's theory of the mass function [1] was done for $t = 1$. It is useful to introduce the correct dependence on the time of $N(\Delta a, t)$. This is done by nondimensionalization: we multiply $N(\Delta a, t)$ by the coalescence length $l_c \sim t^{1/(1-h)}$ and we divide the length Δa by l_c. We thereby obtain:

$$N(\Delta a, t) \sim (\Delta a)^{-h}/t. \tag{18}$$

For the standard Devil's staircase, the length of the Eulerian interval l was chosen as the inverse of the mean number (per unit length) of Lagrangian intervals for the nth generation. For the Burgers case, taking $\Delta a = 2^{-n}$ and using (18), this mean number is $2^{nh}/t$. This is of course also the mean number of shocks per unit (Eulerian) length with a mass of the order of $\Delta a = 2^{-n}$. Hence, the corresponding Eulerian interval, the mean distance between two such successive shocks, is

$$l \sim t\, 2^{-nh} = t(\Delta a)^h. \tag{19}$$

This may also be interpreted as the length of an Eulerian interval such that the mean number of shocks having a Lagrangian length between 2^{-n} and $2^{-(n+1)}$ is order unity.

We can now just repeat for the Burgers problem essentially the same argument as developed for the standard Devil's staircase, to obtain:

$$M_q(l, t) \sim \frac{t}{2^{nh}} \sum_{m=0}^{m=n} N(2^{-m}, t)[2^{-m}]^q \sim \frac{l}{t} \sum_{m=0}^{m=n} 2^{mh}[2^{-m}]^q, \tag{20}$$

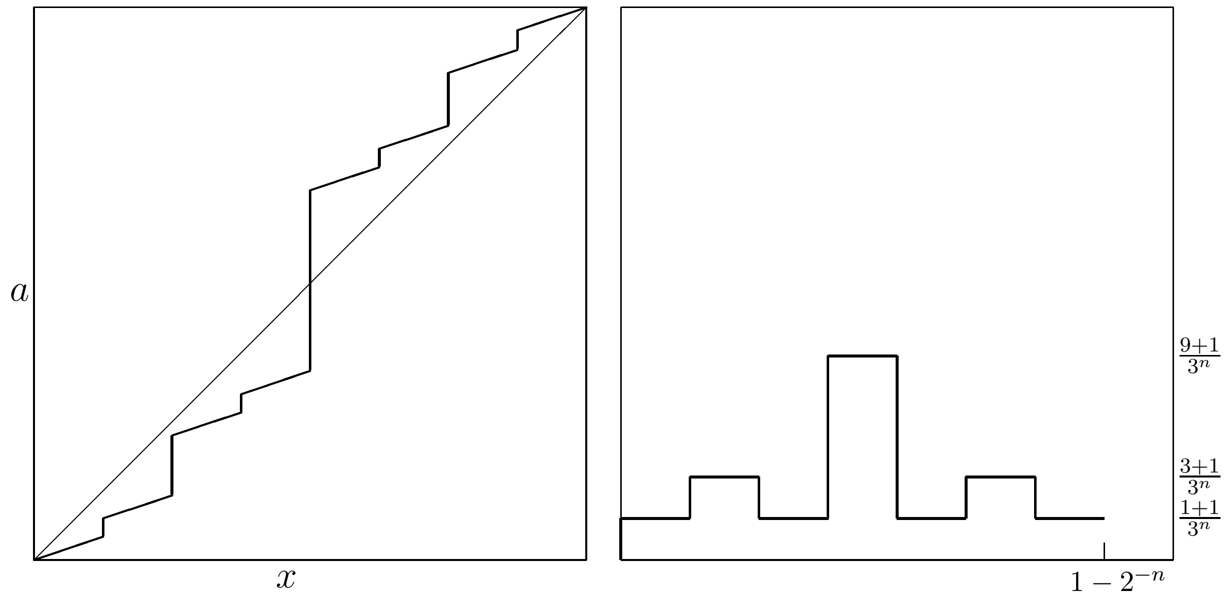

Figure 1: Inverse Devil's staircase and the corresponding increment function

where the upper limit n is given in terms of l by (19).

As before, depending on whether $q > q_\star = h$ or $0 \le q < q_\star$, the sum is dominated by its first or its last term. Hence, we find, in the former case, that $M_q(l,t) \propto l^1$ and, in the latter case, that $M_q(l,t) \propto l^{q/h}$, so that $\tau_q = 1$ for $q > h$ and $\tau_q = q/h$ for $0 \le q < h$,. This establishes the bifractality.

4 Bifractality of shocks: rigorous theory

Let $a(x)$ be a (random) nondecreasing function defined on a finite interval $I \subset \mathbb{R}$. Denote by $M_q(a(\cdot), l)$ the space averaged value of the q-th power of the increment:

$$M_q(a(\cdot), l) := \frac{1}{|I| - l} \int_{I-(l)} [a(x+l) - a(x)]^q \, dx, \tag{21}$$

where $I - (l)$ means that the integration is over the interval I with the exception of an interval of length l at the right end of the interval I, where the increment is not well defined.

Our aim is to study asymptotic properties as $l \to 0$ of the mathematical expectation (mean value) of $M_q(a(\cdot), l)$.

In Section 4.1 we shall derive estimates at first for the inverse of the standard Devil's staircase and then for a general function of this type, assuming that we know the statistical properties of N_k, the number of shocks of magnitude of order 2^{-k}.

In Section 4.2 we prove inequalities for N_k for the case of the solution of Burgers equation with a Brownian initial velocity.

4.1 Estimate of $M_q(a(\cdot), l)$

Let $a(x)$ be the inverse Devil's staircase function on the interval $I = [0, 1]$ and let a_n be its approximation constructed in the same way as for the n-th step of the construction of the standard Cantor set (see Fig. 1). To calculate the functional $M_q(a(\cdot), l)$ we calculate at first the sequence of approximations $M_q(a_m(\cdot), 2^{-n})$, $m \geq n$ for the functions a_m and then investigate its behavior as $n \to \infty$.

For any fixed $n \leq m$ the increment function $a_m(x+2^{-n}) - a_m(x)$ is a piecewise constant function with 2^k pieces of length 2^{-n} of magnitude $3^{-k} + 3^{-n}$ for $k = 0, 1, \ldots, n$ (see Fig. 1). Therefore,

$$\begin{aligned} M_q(a_m(\cdot), 2^{-n}) &:= \frac{1}{1-2^{-n}} \int_0^{1-2^{-n}} [a_m(x+2^{-n}) - a_m(x)]^q \, dx \\ &= \frac{1}{1-2^{-n}} \sum_{k=0}^{n-1} 2^{-n} 2^k \left(3^{-(k+1)} + 3^{-n}\right)^q \\ &= \frac{1}{1-2^{-n}} \sum_{k=0}^{n-1} 2^{-n} 2^k 3^{-(k+1)q} + \frac{1}{1-2^{-n}} \sum_{k=0}^{n-1} 2^{-n} 2^k 3^{-(k+1)q} \cdot R_k, \end{aligned} \tag{22}$$

where

$$R_k := (1 + 3^{-n+k+1})^q - 1 \leq \begin{cases} 2q \, 3^{-n+k+1}, & \text{if } q > 1; \\ 3^{-n+k+1}, & \text{if } q \leq 1; \end{cases} . \tag{23}$$

The first term in (22) can be calculated as follows:

$$\frac{1}{1-2^{-n}} \sum_{k=0}^{n-1} 2^{-n} 2^k 3^{-(k+1)q} = \frac{2^{-n} 3^{-q}}{1-2^{-n}} \sum_{k=0}^{n-1} \left(\frac{2}{3^q}\right)^k = \frac{2^{-n} 3^{-q}}{1-2^{-n}} \frac{1 - \left(\frac{2}{3^q}\right)^n}{1 - \frac{2}{3^q}}, \tag{24}$$

while the second term becomes negligible compared to the first one as $n \to \infty$.

Consider now a more general function $a : I \to \mathbb{R}$ of this type. For each integer $m > 0$ we construct an approximation $a_m(x)$, which is a piecewise linear function with N_k jumps of magnitude $h_k \in [2^{-k-1}, 2^{-k})$ for $0 \leq k < m$. Then the integral of the corresponding increment function with the increment 2^{-n} may be estimated from above by the integral over a piecewise constant function with 2^k pieces of length 2^{-n} of magnitude $2^{-k} + 2^{-n}$ for $0 \leq k < n$. There is a similar estimate from below (with the magnitude $2^{-k-1} + 2^{-n}$). Notice that our estimates do not depend on m, provided $m \geq n$.

Lemma 4.1 *Let the function a_m be as above and let us assume that $C^{-1}2^{k/2} \leq N_k \leq C\,2^{k/2}$. Then*

$$\frac{C^{-1}\,2^{1/2}}{|I|-2^{-n}}\,2^{-n}\frac{1-2^{-n(q-1/2)}}{1-2^{-(q-1/2)}} \leq M_q(a_m(\cdot),2^{-n}) \leq \frac{C\,2^{-(q-1/2)}}{|I|-2^{-n}}\,2^{-n}\frac{1-2^{-n(q-1/2)}}{1-2^{-(q-1/2)}}.$$

Proof. In the same way as we did it for the inverse of the standard Devil's staircase we can estimate $M_q(a_m(\cdot),2^{-n})$ as follows:

$$\begin{aligned} M_q(a_m(\cdot),2^{-n}) &:= \frac{1}{|I|-2^{-n}}\int_{I-(2^{-n})}[a_m(x+2^{-n})-a_m(x)]^q\,dx \\ &\leq \frac{2^{-n}}{|I|-2^{-n}}\sum_{k=0}^{n-1}N_{k+1}\,(h_{k+1}+2^{-n})^q \\ &\leq \frac{2^{-n}C}{|I|-2^{-n}}\sum_{k=0}^{n-1}2^{(k+1)/2}(2^{-(k+1)}+2^{-n})^q \\ &= \frac{C\,2^{-n}2^{-(q-1/2)}}{|I|-2^{-n}}\sum_{k=0}^{n-1}2^{-k(q-1/2)}(1+R_k), \end{aligned} \tag{25}$$

where

$$R_k := (1+2^{-n+k+1})^q - 1 \leq \begin{cases} 2q2^{-n+k+1}, & \text{if } q>1; \\ 2^{-n+k+1}, & \text{if } q\leq 1; \end{cases}. \tag{26}$$

In (25), the term involving R_k becomes negligible compared to the other one as $n\to\infty$ and we are left with :

$$\frac{C\,2^{-(q-1/2)}}{|I|-2^{-n}}\,2^{-n}\frac{1-2^{-n(q-1/2)}}{1-2^{-(q-1/2)}}.$$

The estimate from below is obtained similarly. ∎

Remark 4.2 *To calculate $M_q(a(\cdot),l)$ for a random function $a(x)$ notice, that our estimates depend on N_k linearly. Therefore inequalities for the mathematical expectation of N_k are enough to prove the statement of Lemma 4.1 for the mathematical expectation of $M_q(a(\cdot),l)$.*

4.2 Estimate of N_k.

Let $\xi_\omega(x)$ be a realization of the standard Brownian motion on some finite interval I, and let $\mathcal{C}_w$ be the convex hull of the realization $w(y) := \int_o^y(\xi_\omega(x)+x)\,dx$. We denote by S the set of end points of straight segments (corresponding to shocks) of $\mathcal{C}_w$. This

set is a closed set of zero Lebesgue measure. For any fixed integer $k \geq 0$, we introduce the following notation:

$$N_k := \#\{\Delta_j : 2^{-k-1} \leq |\Delta_j| < 2^{-k}\}, \tag{27}$$

where $\cup_j \Delta_j = I - S$ and Δ_j are intervals, whose end points lies in the set S.

Let us fix two points $x_1, x_2 \in I$ and a small number $0 < \delta \ll 1$. Consider two intervals $I_1 := [x_1 - |x_2 - x_1|\delta, x_1 + |x_2 - x_1|\delta)$ and $I_2 := [x_2 - |x_2 - x_1|\delta, x_2 + |x_2 - x_1|\delta)$. We shall denote by $\zeta(I_1, I_2)$ the indicator function of the event that $\mathcal{C}_w$ has a straight segment whose endpoints lie inside the intervals I_1, I_2, respectively.

Lemma 4.3 (Sinai) *There is a constant $C_0 = C_0(\delta) > 0$ such that*

$$C_0^{-1}|x_2 - x_1|^{1/2} \leq \langle \zeta(I_1, I_2) \rangle \leq C_0 |x_2 - x_1|^{1/2}.$$

Lemma 4.4 *There is a constant $C > 0$ such that $C^{-1} 2^{k/2} \leq \langle N_k \rangle \leq C\, 2^{k/2}$ for all k large enough.*

Proof. Fix k and choose a sufficiently large m. Decompose the segment I onto equal intervals $I_j := [c_j^-, c_j^+)$, indexed from the left to the right, of length $2^{-k}/m$ and consider the pairs I_i, I_j such that $m/2 \leq j - i \leq m + 2$. Then

$$2^{-k-1} \leq (j - i)2^{-k}/m \leq 2^{-k} - 2 \cdot 2^{-k}/m, \tag{28}$$

which means that for any pair of points $x \in I_i$ and $y \in I_j$ we have $2^{-k-1} \leq |x - y| < 2^{-k}$. Therefore,

$$N_k = \sum_{ij} \zeta(I_i, I_j). \tag{29}$$

By Lemma 4.3 the mathematical expectation of ζ can be estimated as follows

$$C_0^{-1} \sqrt{1/2}\, 2^{-k/2} \leq \langle \zeta(I_i, I_j) \rangle \leq C_0 \sqrt{1 - 2/m}\, 2^{-k/2}. \tag{30}$$

Thus, summing over all pairs of indices, we have:

$$C^{-1}\, 2^{k/2} \leq \langle N_k \rangle \leq C_1\, m^2\, 2^k\, 2^{-k/2} = C\, 2^{k/2}, \tag{31}$$

which completes the proof. ∎

M_q as a function of l

Figure 2: Moments of order q, as labeled, of Lagrangian increment *vs.* separation l for Brownian initial velocity ($h = 1/2$) at $t = 0.5$. The simulations used periodic conditions of unit size with a mesh $\epsilon = 2^{-22}$. Notice the conspicuous, but spurious, scaling with unity exponent at small separations.

Local scaling exponent for $h = 1/2$

Figure 3: Local scaling exponent obtained as the logarithmic derivative of $M_q(l)$ with respect to l. Same conditions as Fig. 2.

5 Simulations and spurious scaling régime

The numerical strategy for solving Burgers equation with fractional Brownian initial velocity has been described in detail in Ref. [3] (Section 5). Let us just recall some key points here. In the simulations, it is necessary to introduce both a large-scale and a small-scale cutoff. We find it convenient to work with periodic velocity fields and to set the spatial period to unity. Then, the scales accessible are clearly restricted to the range $\epsilon < l < 1$, where $\epsilon = 1/N$ is the inverse of the number of grid points in the simulation. In Ref. [3] up to 2^{20} grid points were used. For reasons which will become clear, we had here to work with even higher resolution, using $N = 2^{22}$. Moments were calculated by averaging over space and over about 1200 realizations. The constant C appearing in (7) is always taken unity.

Fig. 2 gives log-log plots of the moments $M_q(l)$ for $h = 1/2$ and 17 values of the exponent q varying linearly between zero and two. The graphs appear to all have the same unity slope at small l's and a different q-dependent slope at large l's. Fig. 3 shows the logarithmic derivative of $M_q(l)$ with respect to l, a measure of the local scaling exponent. It is seen that below a separation l of about 10^{-4} all the exponents τ_q go to unity, while at very large separations they appear to approach the value $\tau_q = q$ (at least for q's up to one). The latter result is an immediate consequence of the fact that, at separations much larger than the coalescence length l_c, the Lagrangian map is very close to the identity. Hence, $M_q(l) \simeq l^q$.

We shall now show that, when $0 \le q \le q_\star = h$, the former result $\tau_q = 1$ is a numerical *artifact* due to a spurious discretization effect, affecting all separations such that $l < l_{\mathrm{sp}} \sim t\epsilon^h$, where ϵ is the numerical mesh. By (19), for a given l, the

Local scaling exponent for $h = 3/4$

Figure 4: Same as Fig. 3, but for $h = 3/4$. Notice that the region of spurious scaling $\tau_q = 1$ has shifted to smaller separations and that a region with $\tau_q = q/h$ for small q becomes visible.

dominant contribution to $M_q(l)$ should come from those shocks with a Lagrangian length $\Delta a = 2^{-n}$ such that,

$$l \sim t\, 2^{-nh} \sim t\,(\Delta a)^h. \tag{32}$$

Clearly, no shocks can be represented which have $\Delta a < \epsilon$, where ϵ is the numerical mesh. This gives a cutoff in l at

$$l_{\rm sp} \sim t\,\epsilon^h. \tag{33}$$

For $l < l_{\rm sp}$, the numerically measured Lagrangian increment will typically take only two values: ϵ with probability $l/l_{\rm sp}$ and zero with probability $1 - l/l_{\rm sp}$. Hence,

$$M_q(l) \sim \epsilon^q l/l_{\rm sp}, \tag{34}$$

which implies $\tau_q = 1$. This is the spurious scaling announced.

Thus, the scaling régime with $\tau_q = q/h$, discussed in the theoretical part of this paper, can be observed only at scales such that

$$l_{\rm sp} \sim t\epsilon^h \ll l \ll l_c \sim t^{1/(1-h)}, \tag{35}$$

where

$$l_c \sim t^{1/(1-h)}. \tag{36}$$

is the coalescence length [3].

In practice (35) is a strong constraint: if we also want to avoid contaminations of τ_q due to finite box size, we should take $l_c \sim t^{1/(1-h)}$ significantly smaller than unity. Hence, for $h = 1/2$, the range of nonspurious separations defined by (35) will be too small to be visible unless we work at extremely high resolutions (very small ϵ's).

Inspection of (35) reveals, however, another strategy: we can increase h and thereby push the spurious range to smaller separations. Since the fractional Brownian properties of the velocity disappear at $h = 1$, we chose $h = 3/4$ as a trade-off. Fig. 4 shows the same result as in Fig. 3 but, now, for $h = 3/4$. We observe that the region of spurious scaling has now been pushed below $l \approx 10^{-5}$ and that for $0 < q < q_\star = h = 3/4$ a kind of plateau near $\tau_q = q/h$ is seen at the smallest separations not affected by spurious scaling. Measuring directly the exponents τ_q in this range produces Fig. 5 where τ_q is plotted *vs.* the exponent q, both for $h = 1/2$

Phase transition

Figure 5: Scaling exponent measured at the smallest separation not affected by spurious scaling, plotted *vs.* the order q for $h = 1/2$ and $h = 3/4$, as labeled. The piecewise linear graphs represent theoretical predictions.

and $h = 3/4$. The comparison with our theoretical prediction of Section 3 (the thick straight lines) is now satisfactory, the only remaining discrepancies being caused by unavoidable finite size effects which soften the phase transition.

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